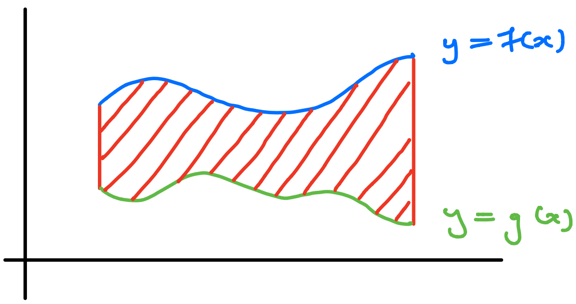
The Area Between Two Curves

f, g - continuous functions on $[a,b]$.

$f(x) \geq g(x)$ for all x in $[a,b]$

E.g.



Q: What is Area (▨) ?

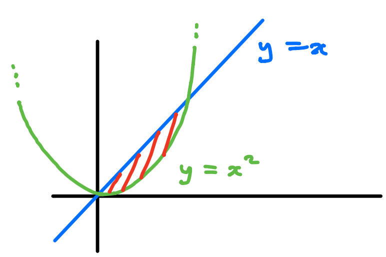
Assume $f(x), g(x) \geq 0$ on $[a,b]$

$$\Rightarrow \quad \text{Area}(▨) = \begin{pmatrix}\text{Area under } y=f(x) \\ \text{between } a \text{ and } b\end{pmatrix} - \begin{pmatrix}\text{Area under } y=g(x) \\ \text{between } a \text{ and } b\end{pmatrix}$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx$$

Example What is the finite region enclosed by $y = x$ and $y = x^2$?



Area (▨) = ?

Need to find intersection points : $x = x^2 \Rightarrow x - x^2 = 0$

$\Rightarrow x(1-x) = 0$

$\Rightarrow x = 0$ or 1

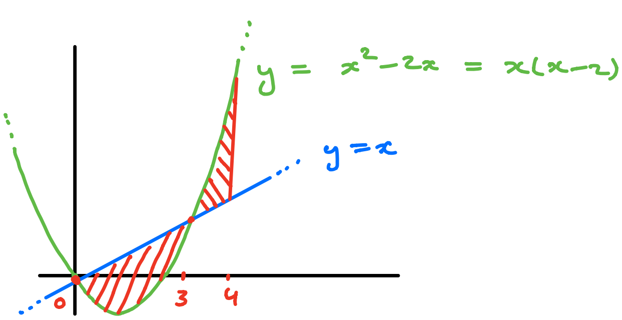
$$\Rightarrow \text{Area}(\text{▨}) = \int_0^1 (x - x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{6}$$

Fact : This formula also works even when $f(x)$ and $g(x)$ are not always positive on $[a,b]$.

$$\boxed{\begin{array}{l}\text{Conclusion : } f, g \text{ continuous on } [a,b] \text{ and } f(x) \geq g(x) \text{ on } [a,b] \Rightarrow \\ \quad \text{Area between } y=f(x) \text{ and } y=g(x) \text{ from } a \text{ to } b = \displaystyle\int_a^b (f(x) - g(x))\,dx\end{array}}$$

We must be careful when $y = f(x)$ and $y = g(x)$ cross each other on $[a,b]$.

Example What is the area of the region enclosed by $y = x^2 - 2x$ and $y = x$ on $[0,4]$?



Intersections :

$x^2 - 2x = x \Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0 \Rightarrow x = 0$ or 3

$$\Rightarrow \quad \text{Area}(\ ) = \int_0^3 x - (x^2 - 2x)\, dx = \int_0^3 3x - x^2\, dx$$

$$= \frac{3}{2}x^2 - \frac{x^3}{3} \Big|_0^3 = \frac{27}{2} - 9$$

$$\text{Area}(\ ) = \int_3^4 (x^2 - 2x) - x\, dx = \int_3^4 x^2 - 3x\, dx$$

$$= \frac{1}{3}x^3 - \frac{3}{2}x^2 \Big|_3^4 = \left(\frac{64}{3} - 24 - 9 + \frac{27}{2} \right)$$

$$\Rightarrow \quad \begin{array}{l}\text{Area enclosed} \\ \text{between 0 and 4}\end{array} = \frac{27}{2} - 9 + \left(\frac{64}{3} - 24 - 9 + \frac{27}{2} \right)$$

$$= \frac{19}{3}$$

continuous

Strategy For Finding area of region enclosed by $y = f(x)$ and $y = g(x)$

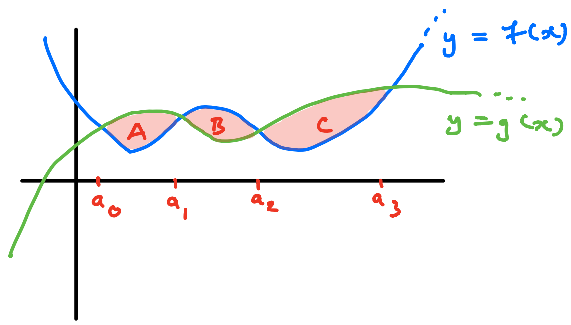
1/ Set $f(x) = g(x)$ and find all possible solutions. These give possible crossing points

2/ Between sucessive crossing calculate area enclosed using $\int f(x) - g(x)\, dx$ or $\int g(x) - f(x)\, dx$

3/ Sum each area

4/ If there are endpoints a and b, restrict to interval $[a, b]$.

Basic Picture :



= Region enclosed by $y = f(x)$ and $y = g(x)$

$$\text{Area}(A) = \int_{a_0}^{a_1} g(x) - f(x) \, dx$$

$$\text{Area}(B) = \int_{a_1}^{a_2} f(x) - g(x) \, dx$$

$$\text{Area}(C) = \int_{a_2}^{a_3} g(x) - f(x) \, dx$$

Pro Tip : If you get f and g wrong way around on a segment just take negative of your answer. This means you don't really need to draw graphs.

Example What is the area of the region enclosed by $y = x^3 - x^2 + x + 1$ and $y = 2x^2 - x + 1$?

$$x^3 - x^2 + x + 1 = 2x^2 - x + 1 \Rightarrow x^3 - 3x^2 + 2x = 0$$

$$\Rightarrow x(x^2 - 3x + 2) = 0 \Rightarrow x(x-2)(x-1) = 0 \Rightarrow x = 0, 1, 2$$

$\Rightarrow$ There will be two enclosed regions A and B.

Region A Let's guess

$$\int_0^1 \left((x^3 - x^2 + x + 1) - (2x^2 - x + 1) \right) dx = \int_0^1 x^3 - 3x^2 + 2x \, dx$$

$$= \frac{1}{4}x^4 - x^3 + x^2 \Big|_0^1 = \frac{1}{4} - 1 + 1 = \frac{1}{4} > 0$$

This must be correct as it's positive.

$$\Rightarrow \text{Area}(A) = \frac{1}{4}$$

Region B :

$$\int_1^2 \left((x^3-x^2+x+1)-(2x^2-x+1)\right)dx = \int_1^2 x^3-3x^2+2x\,dx$$

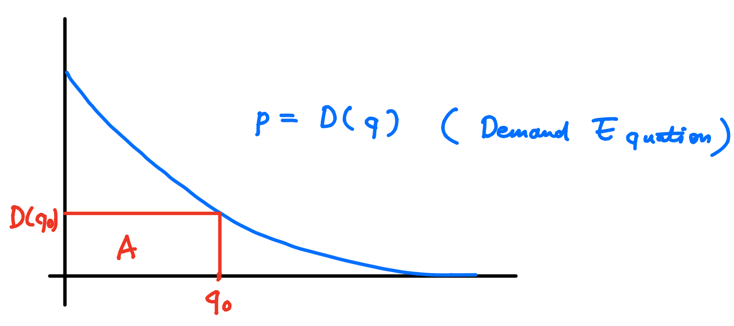
$$= \frac{1}{4}x^4 - x^3 + x^2 \Big|_1^2 = (4-8+4) - \left(\frac{1}{4}-1+1\right) = \frac{-1}{4}$$

This cannot be correct as it is negative. However all we have to do is change sign $\Rightarrow$ Area (B) $= \frac{1}{4}$

$\Rightarrow$ Area of region enclosed by
$y = x^3 - x^2 + x + 1$ and
$y = 2x^2 - x + 1$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Consumer Surplus

Q, : Can we calculate the total amount of money consumers are willing to pay for a product.

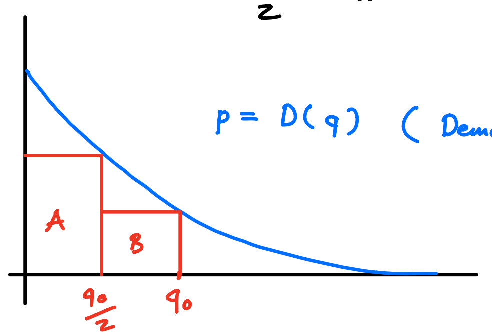


Maximum revenue from selling q_0 units in one batch at one price. $= q_0 \times D(q_0) = $ Area (A)

Problem: Some people are willing to pay more than p_0

Solution: Divide q_0 units into multiple batches and sell at different prices.

2 Batches of $\frac{q_0}{2}$ units:

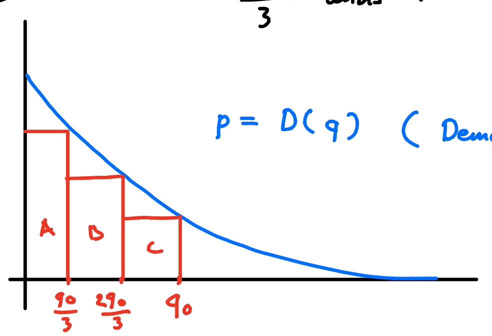


Area (A) = Maximum revenue from selling first batch of $\frac{q_0}{2}$

Area (B) = Maximum revenue from selling second batch of $\frac{q_0}{2}$.

Area (A) + Area (B) = Maximum revenue from selling 2 batches of $\frac{q_0}{2}$ units at different prices.

3 Batches of $\frac{q_0}{3}$ units:



Area (A) = Maximum revenue from selling first batch of $\frac{q_0}{3}$

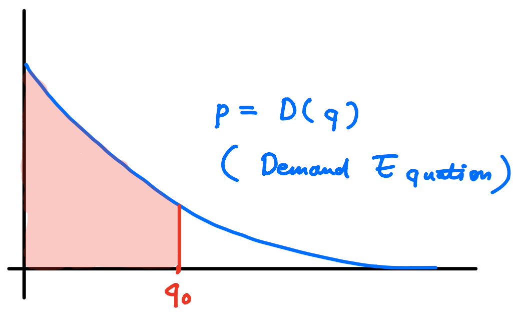
Area (B) = Maximum revenue from selling second batch of $\frac{q_0}{3}$.

Area (C) = Maximum revenue from selling second batch of $\frac{q_0}{3}$.

Area(A) + Area(B) + Area(C) = Maximum revenue from selling 3 batches of $\frac{q_0}{3}$ units at different prices.

The greater the number of batches the closer the revenue is to the total amount consumers are willing to pay for q_0 units.

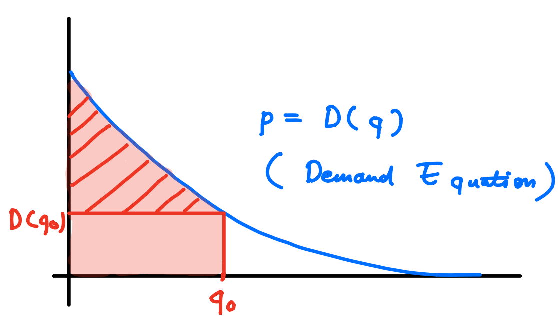
$\Rightarrow$



Area ( ) = Total amount consumers are willing to pay for q_0 units

<u>Conclusion</u> : Total amount consumers are willing to pay for q_0 units $= \int_0^{q_0} D(q)\,dq$

Consumer supplus from sale of q_0 units at one price = Total amount consumers are willing to pay for q_0 units − Maximum revenue from selling q_0 units in <u>one</u> batch at one price.



$\Rightarrow$ Area ( ) = Consumer Surplus

Conclusion

Consumer surplus from sale of q_0 units at one price $= \displaystyle\int_0^{q_0} \left(D(q) - D(q_0) \right) dq$

Remark Often q_0 is chosen to be the equilibrium quanity.